A LINEARIZED APPROACH TO WORST-CASE DESIGN IN SHAPE OPTIMIZATION

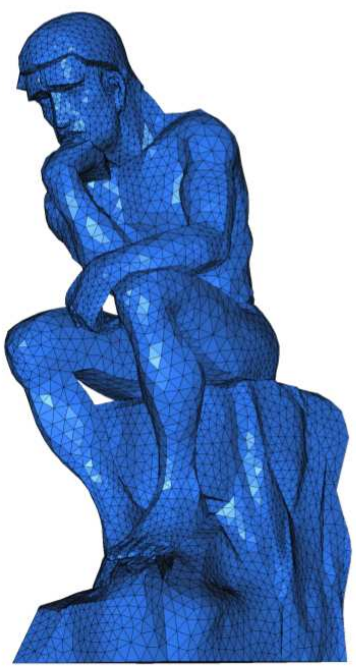
Grégoire ALLAIRE
CMAP, Ecole Polytechnique

Results obtained in collaboration with Ch. Dapogny (Rutgers University, formerly LJLL UPMC and Renault).

Reims, 4 novembre 2014.

CONTENTS

RODIN project



Ecole Polytechnique,

UPMC, INRIA,

Renault, EADS,

ESI group, etc.

-I- INTRODUCTION

Shape optimization : minimize an objective function over a set of admissibles shapes Ω (including possible constraints)

$$\inf_{\Omega \in \mathcal{U}_{ad}} J(\Omega)$$

The objective function is evaluated through a partial differential equation (state equation)

$$J(\Omega) = \int_\Omega j(u_\Omega)\, dx$$

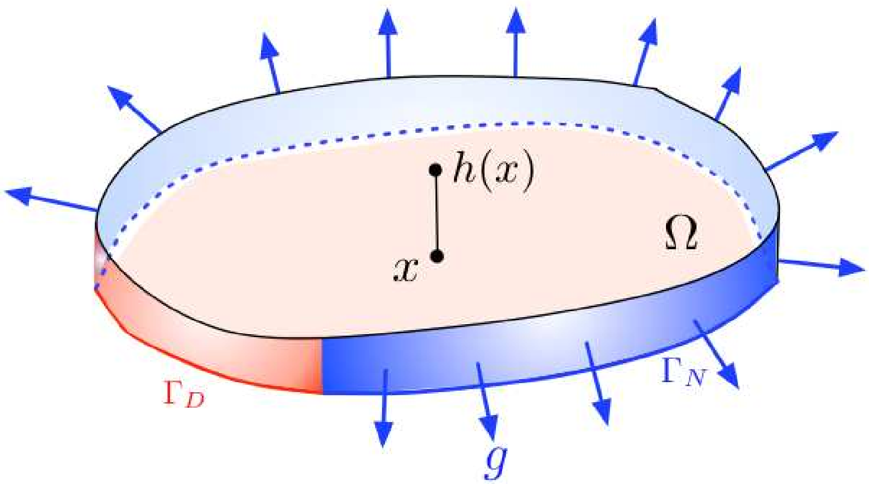
where u_Ω is the solution of

$$PDE(u_\Omega) = 0 \quad \text{in} \quad \Omega$$

Thickness optimization : the shape is parametrized by its thickness h (a coefficient in the p.d.e.).

Geometric optimization : the boundary of Ω is varying.

Thickness optimization



Mid-plane $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$.

Thickness of the plate $h(x) : \Omega \to [h_{\min}, h_{\max}]$ with $h_{\max} > h_{\min} > 0$.

Thickness optimization (Ctd.)

For given applied loads $g : \Gamma_N \to \mathbb{R}^d$, $f : \Omega \to \mathbb{R}^d$, the displacement $u : \Omega \to \mathbb{R}^d$ is the solution of

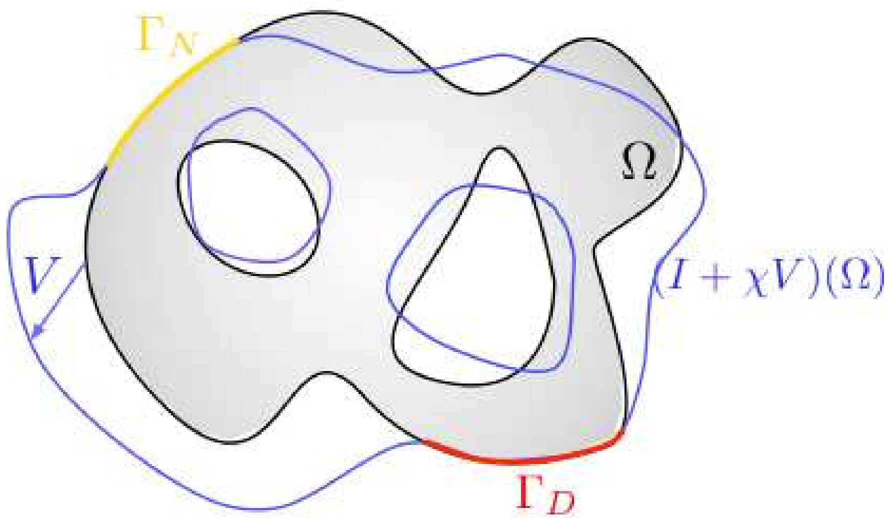
$$\begin{cases} -\operatorname{div}\left(hA\, e(u)\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(hA\, e(u)\big)n = g & \text{on } \Gamma_N \end{cases}$$

with the strain tensor $e(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right)$, the stress tensor $\sigma = hAe(u)$, and A an homogeneous isotropic elasticity tensor.

Typical objective function: the compliance

$$J(h) = \int_\Omega f \cdot u\, dx + \int_{\Gamma_N} g \cdot u\, dx,$$

Geometric optimization



Shape $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$, where Γ_D and Γ_N are fixed.
Only Γ is optimized (free boundary).

Geometric optimization (Ctd.)

For given applied loads $g : \Gamma_N \to \mathbb{R}^d$, $f : \Omega \to \mathbb{R}^d$, the displacement $u : \Omega \to \mathbb{R}^d$ is the solution of

$$\begin{cases} -\operatorname{div}(A\,e(u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n = g & \text{on } \Gamma_N \\ \big(A\,e(u)\big)n = 0 & \text{on } \Gamma \end{cases}$$

Typical objective function: the compliance

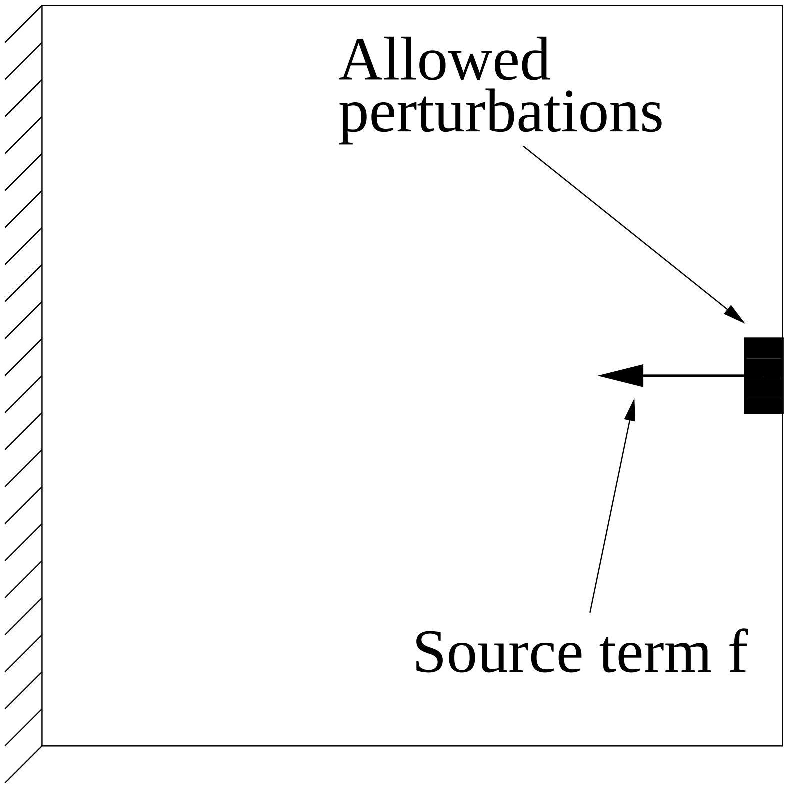
$$J(\Omega) = \int_\Omega f \cdot u\,dx + \int_{\Gamma_N} g \cdot u\,dx,$$

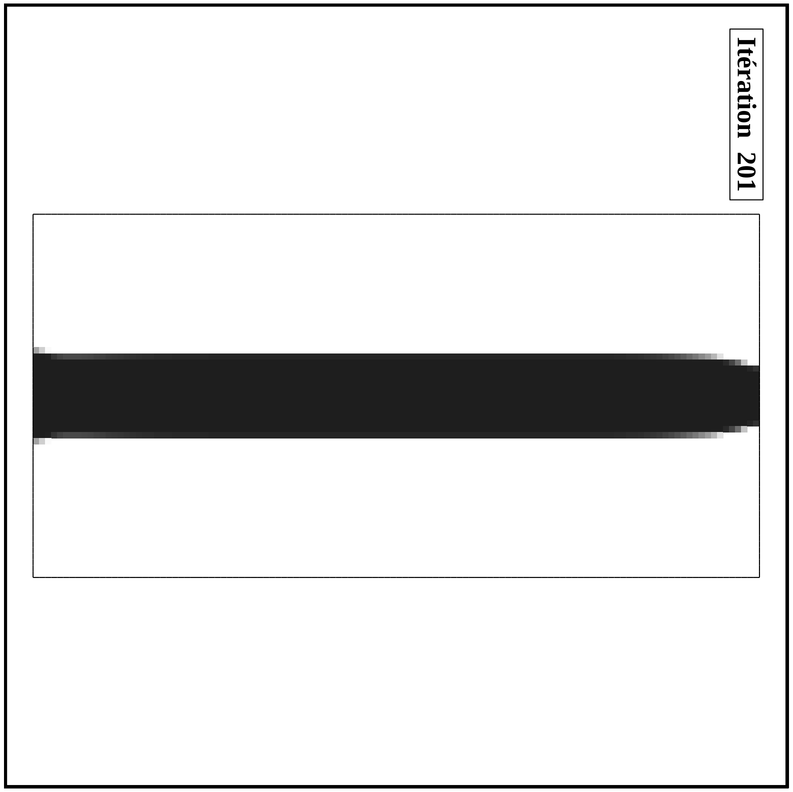
Uncertainties

- ☞ location, magnitude and orientation of the body forces or surface loads
- ☞ elastic material's properties
- ☞ geometry of the shape (thickness or boundary)

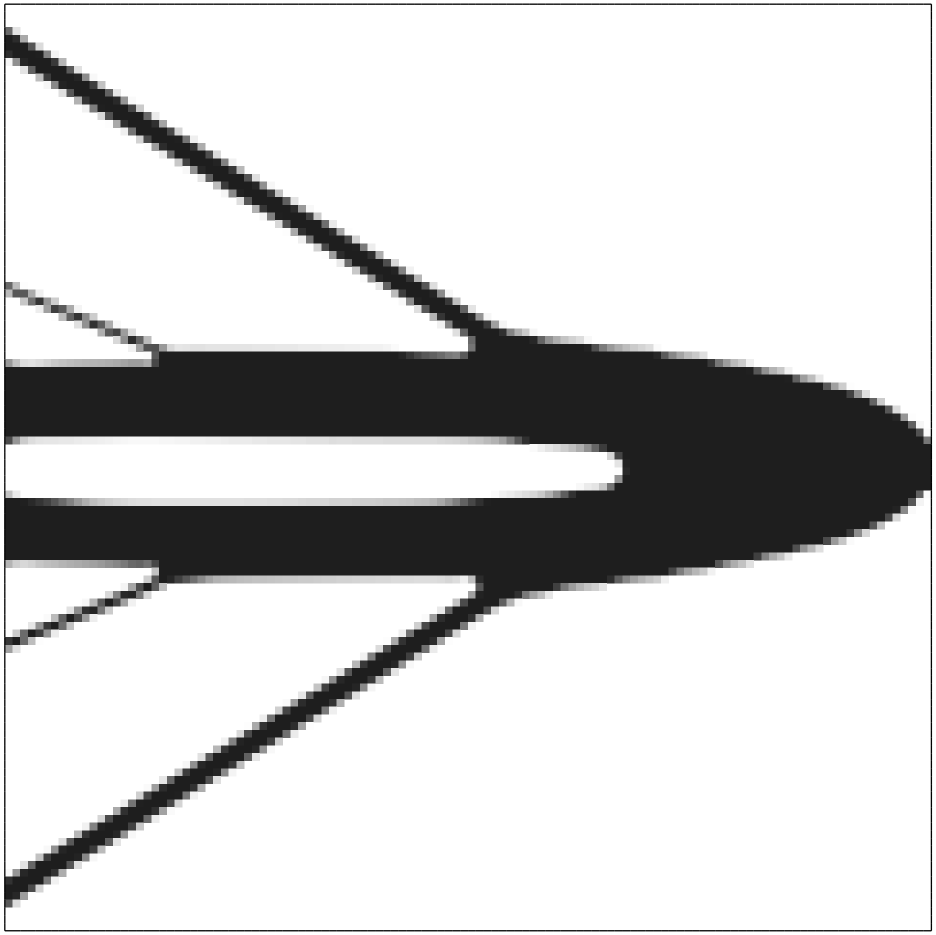
Crucial issue: optimal structures are so optimal for a given set of loads that they cannot sustain a different load !

Example: minimal weight and minimal compliance





Optimal design with load uncertainties



State of the art: many works !

☞ Probabilistic approach (Ben-Tal et al. 97, Choi et al. 2007, Frangopol-Maute 2003, Kalsi et al. 2001...)

- Monte-Carlo methods
- Polynomial chaos, Karhunen-Loève expansions...
- First-Order Reliability-based Methods (FORM)

☞ Various objectives or goals:

- Minimization of expected value or mean
- Worst case desing
- Minimal failure probability

☞ Special cases with simplifications:

- Robust compliance: Cherkaev-Cherkaeva (1999, 2003), de Gournay-Allaire-Jouve (2008).
- Mean expectation of compliance: Alvarez-Carrasco 2005, Dunning-Kim 2013...

☞ Present work: two main ideas

- worst case optimization (min-max problem),
- linearization for small uncertainties (similar idea in Babuska-Nobile-Tempone 2005).

Worst case design

Example in the case of force uncertainties.

The force is the sum $f+\xi$ where f is known and ξ is unknown.

The only information is the location of ξ and its maximal magnitude $m>0$ such that $\|\xi\| \leq m$.

We replace the standard objective function $J(\Omega, f+\xi)$ by its worst case version $\mathcal{J}(\Omega, f)$.

Worst case design optimization problem:

$$\min_{\Omega} \mathcal{J}(\Omega, f) = \min_{\Omega} \max_{\|\xi\| \leq m} J(\Omega, f+\xi)$$

-II- ABSTRACT (AND FORMAL) SETTING

- ☞ Designs $h \in \mathcal{H}$
- ☞ State equation $\mathcal{A}(h)u(h) = b$ with a linear operator $\mathcal{A}(h)$
- ☞ Perturbations $\delta \in \mathcal{P}$ in a Banach space $\mathcal{P}$
- ☞ Assume for simplicity that only b (not $\mathcal{A}$) depends on δ
- ☞ Perturbed state equation $\mathcal{A}(h)u(h,\delta) = b(\delta)$
- ☞ Worst case objective function

$$\mathcal{J}(h) = \sup_{\substack{\delta \in \mathcal{P} \\ ||\delta||_{\mathcal{P}} \leq m}} J(u(h,\delta))$$

- ☞ Goal

$$\inf_{h \in \mathcal{H}} \mathcal{J}(h)$$

Linearization

Assume that the perturbations are small, i.e., $m << 1$.

☞ Unperturbed case $\delta = 0$, $u(h) = u(h,0)$

☞ Derivative of the state equation

$$\mathcal{A}(h)\frac{\partial u}{\partial \delta}(h,0) = \frac{db}{d\delta}(0)$$

☞ Linearization of the worst-case objective function

$$\mathcal{J}(h) \approx \widetilde{\mathcal{J}}(h) = \sup_{\substack{\delta \in \mathcal{P} \\ ||\delta||_{\mathcal{P}} \leq m}} \left(J(u(h)) + \frac{dJ}{du}(u(h))\frac{\partial u}{\partial \delta}(h,0)(\delta) \right)$$

Since the right hand side is linear in δ we deduce

$$\widetilde{\mathcal{J}}(h) = J(u(h)) + m \left|\left| \frac{dJ}{du}(u(h))\frac{\partial u}{\partial \delta}(h,0) \right|\right|_{\mathcal{P}^*}$$

Adjoint approach

The previous formula for $\widetilde{\mathcal{J}}(h)$ is not fully explicit:

$$\widetilde{\mathcal{J}}(h) = J(u(h)) + m \left\| \frac{dJ}{du}(u(h)) \frac{\partial u}{\partial \delta}(h,0) \right\|_{\mathcal{P}^*}$$

Introduce an adjoint state

$$\mathcal{A}(h)^T p(h) = \frac{dJ}{du}(u(h)),$$

from which we deduce

$$\mathcal{A}(h)^T p(h) \cdot \frac{\partial u}{\partial \delta}(h,0) = \mathcal{A}(h) \frac{\partial u}{\partial \delta}(h,0) \cdot p(h) = \frac{dJ}{du}(u(h)) \cdot \frac{\partial u}{\partial \delta}(h,0) = \frac{db}{d\delta}(0) \cdot p(h)$$

Conclusion:

$$\widetilde{\mathcal{J}}(h) = J(u(h)) + m \left\| \frac{db}{d\delta}(0) \cdot p(h) \right\|_{\mathcal{P}^*}$$

Linearized worst-case design

We add to the usual objective function a perturbation term which is proportional to m and to the standard adjoint p:

$$\widetilde{\mathcal{J}}(h) = J(u(h)) + m \left|\left| \frac{db}{d\delta}(0) \cdot p(h) \right|\right|_{\mathcal{P}^*}$$

- ☞ Classical sensitivity approach can be applied to $\widetilde{\mathcal{J}}(h)$
- ☞ The appearance of the adjoint is not a surprise: it is known to measure the sensitivity of the optimal value with respect to the constraint level (or right hand side in the state equation).
- ☞ The entire argument needs to be made rigorous in each specific case.
- ☞ We don't say anything about the existence of optimal designs.
- ☞ We don't prove that optimal designs for $\widetilde{\mathcal{J}}(h)$ are close to those of $\mathcal{J}(h)$.

What remains to be done (in this talk)

Linearized worst-case design optimization:

$$\inf_{h\in\mathcal{H}} \left\{ \widetilde{\mathcal{J}}(h) = J(u(h)) + m \left\| \frac{db}{d\delta}(0) \cdot p(h) \right\|_{\mathcal{P}^*} \right\}$$

where

$$\mathcal{A}(h)u(h) = b(0) \quad \text{and} \quad \mathcal{A}(h)^T p(h) = \frac{dJ}{du}(u(h)),$$

☞ We compute a derivative of $\widetilde{\mathcal{J}}(h)$: it requires two additional adjoints !

☞ We build a gradient-based algorithm.

☞ We test it on various objective functions.

-III- THICKNESS OPTIMIZATION

First case: loading uncertainties.

Given load $f \in L^2(\Omega)^d$. Unknown load $\xi \in L^2(\Omega)^d$ with small norm $\|\xi\|_{L^2(\Omega)^d} \leq m$. Solution $u_{h,\xi}$ of

$$\begin{cases} -\operatorname{div}\left(hA\, e(u_{h,\xi})\right) = f + \xi & \text{in } \Omega \\ u_{h,\xi} = 0 & \text{on } \Gamma_D \\ \big(hA\, e(u_{h,\xi})\big) n = g & \text{on } \Gamma_N \end{cases}$$

Many variants are possible (ξ may be localized, or parallel to a fixed vector, or on Γ_N, etc.)

Given a smooth (+ growth conditions) integrand j, consider

$$J(h,\xi) = \int_\Omega j(\xi, u_{h,\xi})\, dx$$

Worst case design objective function:

$$\mathcal{J}(h) = \sup_{\substack{\xi \in L^2(\Omega)^d \\ ||\xi||_{L^2(\Omega)^d} \leq m}} J(h,\xi)$$

Linearized worst case design objective function:

$$\widetilde{\mathcal{J}}(h) = \sup_{\substack{\xi \in L^2(\Omega)^d \\ ||\xi||_{L^2(\Omega)^d} \leq m}} \left(J(h,0) + \frac{\partial J}{\partial f}(h,0)(\xi) \right)$$

Theorem.

$$\widetilde{\mathcal{J}}(h) = \int_\Omega j(0,u_h)\,dx + m\,||\nabla_f j(0,u_h) - p_h||_{L^2(\Omega)^d}\,,$$

where p_h is the first adjoint state, defined by

$$\begin{cases} -\mathrm{div}(hAe(p_h)) & = & -\nabla_u j(0,u_h) & \text{in } \Omega, \\ p_h & = & 0 & \text{on } \Gamma_D, \\ hAe(p_h)n & = & 0 & \text{on } \Gamma_N. \end{cases}$$

If $\nabla_f j(0,u_h) - p_h \neq 0$ in $L^2(\Omega)^d$, then $\widetilde{\mathcal{J}}$ is Fréchet differentiable

$$\widetilde{\mathcal{J}}'(h)(s) = \int_\Omega \mathcal{D}(u_h,p_h,q_h,z_h)\,s\,dx,$$

with two additional adjoints q_h, z_h and

$$\mathcal{D}(u_h,p_h,q_h,z_h) := Ae(u_h):e(p_h) + m\frac{Ae(u_h):e(z_h) + Ae(p_h):e(q_h)}{2\,||\nabla_f j(0,u_h) - p_h||_{L^2(\Omega)^d}}$$

The second and third adjoint states q_h, z_h are defined by

$$\left\{\begin{array}{rcll} -\mathrm{div}(hAe(q_h)) & = & -2\,(p_h - \nabla_f j(0,u_h)) & \text{in } \Omega, \\ q_h & = & 0 & \text{on } \Gamma_D, \\ hAe(q_h)n & = & 0 & \text{on } \Gamma_N, \end{array}\right.$$

$$\left\{\begin{array}{rcll} -\mathrm{div}(hAe(z_h)) & = & -2\,\nabla_f \nabla_u j(u_h)^T\,(\nabla_f j(u_h) - p_h) - \nabla_u^2 j(u_h) q_h & \text{in } \Omega, \\ z_h & = & 0 & \text{on } \Gamma_D, \\ hAe(z_h)n & = & 0 & \text{on } \Gamma_N. \end{array}\right.$$

Second case: thickness uncertainties.

Given thickness $h \in L^\infty(\Omega)$. Uncertainty $s \in L^\infty(\Omega)$ with $\|s\|_{L^\infty(\Omega)} \leq m$.

$$\begin{cases} -\operatorname{div}\left((h+s)A\,e(u_{h+s})\right) = f & \text{in } \Omega \\ u_{h+s} = 0 & \text{on } \Gamma_D \\ \big((h+s)A\,e(u_{h+s})\big)n = g & \text{on } \Gamma_N \end{cases}$$

Worst case design objective function:

$$\mathcal{J}(h) = \sup_{\substack{s\in L^\infty(\Omega) \\ \|s\|_{L^\infty(\Omega)}\leq m}} \left\{ J(h+s) = \int_\Omega j(u_{h+s})\,dx \right\}$$

Linearized worst case design objective function:

$$\widetilde{\mathcal{J}}(h) = \sup_{\substack{s\in L^\infty(\Omega) \\ \|s\|_{L^\infty(\Omega)}\leq m}} \left(J(h) + \frac{\partial J}{\partial h}(h)(s) \right)$$

Theorem.

$$\widetilde{\mathcal{J}}(h) = \int_\Omega j(u_h)\,dx + m\,||Ae(u_h) : e(p_h)||_{L^1(\Omega)}\,,$$

where p_h is the first adjoint state, defined by

$$\begin{cases} -\mathrm{div}(hAe(p_h)) = -\nabla_u j(u_h) & \text{in } \Omega \\ p_h = 0 & \text{on } \Gamma_D \\ hAe(p_h)n = 0 & \text{on } \Gamma_N \end{cases} .$$

If $E_h := \{x \in \Omega,\ Ae(u_h) : e(p_h) = 0\}$ has zero Lebesgue measure, then $\widetilde{\mathcal{J}}$ is differentiable

$$\widetilde{\mathcal{J}}'(h)(s) = \int_\Omega s\Big(Ae(u_h) : e(p_h) + m\Big(Ae(p_h) : e(q_h) + Ae(u_h) : e(z_h)\Big)\Big)\,dx,$$

with two additional adjoint states q_h, z_h.

NUMERICAL ALGORITHM

1. Initialization of the thickness h_0.
2. Iteration until convergence for $k \geq 1$:
   (a) Computation of u_k and the 3 adjoints p_k, q_k, z_k by solving linearized elasticity problem with the thickness h_k. Evaluation of the gradient $\widetilde{\mathcal{J}}'(h_k)$
   (b) Update of the thickness h_{k+1} by a projected gradient step (to satisfy bounds and volume constraint).

All computations are made with FreeFem++.

Load uncertainties in thickness optimization

Compliance minimization

$$J(h,\xi) = \int_\Omega (f+\xi)\cdot u_{h,\xi}\, dx$$

with a fixed volume constraint

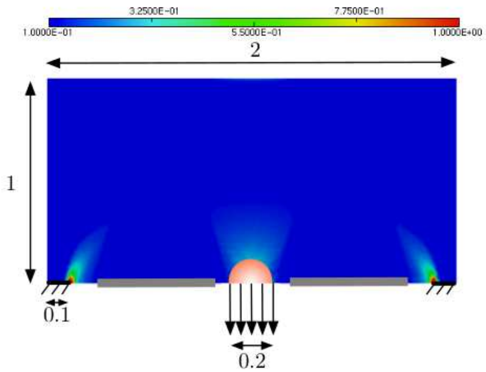
$$\mathrm{Vol}(h) := \int_\Omega h\, dx = 0.7$$

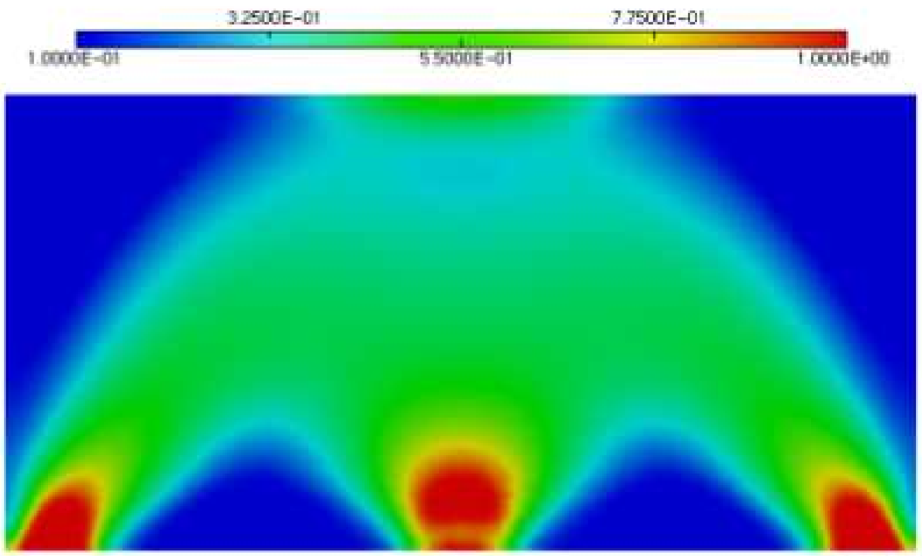
Rectangular 2×1 domain. Bounds $h_{min} = 0.1$ and $h_{max} = 1$.

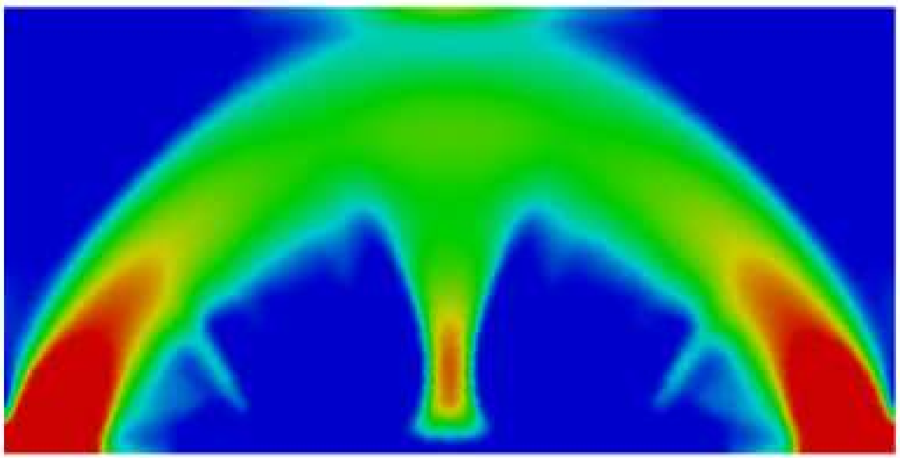
Material properties $E = 1, \quad \nu = 0.3$.

We compute optimal designs for increasing values of m.

Load uncertainties in thickness optimization







-IV- GEOMETRIC OPTIMIZATION

First case: loading uncertainties.

Given load $f \in L^2(\mathbb{R}^d)^d$. Unknown load $\xi \in L^2(\mathbb{R}^d)^d$ with small norm $\|\xi\|_{L^2(\mathbb{R}^d)^d} \leq m$. Solution $u_{\Omega,\xi}$ of

$$\begin{cases} -\operatorname{div}(A\,e(u_{\Omega,\xi})) = f + \xi & \text{in } \Omega \\ u_{\Omega,\xi} = 0 & \text{on } \Gamma_D \\ \big(A\,e(u_{\Omega,\xi})\big)n = g & \text{on } \Gamma_N \\ \big(A\,e(u_{\Omega,\xi})\big)n = 0 & \text{on } \Gamma \end{cases}$$

Many variants are possible (ξ may be localized, or parallel to a fixed vector, or on Γ_N, etc.)

Theorem.

$$\tilde{\mathcal{J}}(\Omega) = \int_\Omega j(0, u_\Omega)\, dx + m||\nabla_f j(0, u_\Omega) - p_\Omega||_{L^2(\Omega)^d},$$

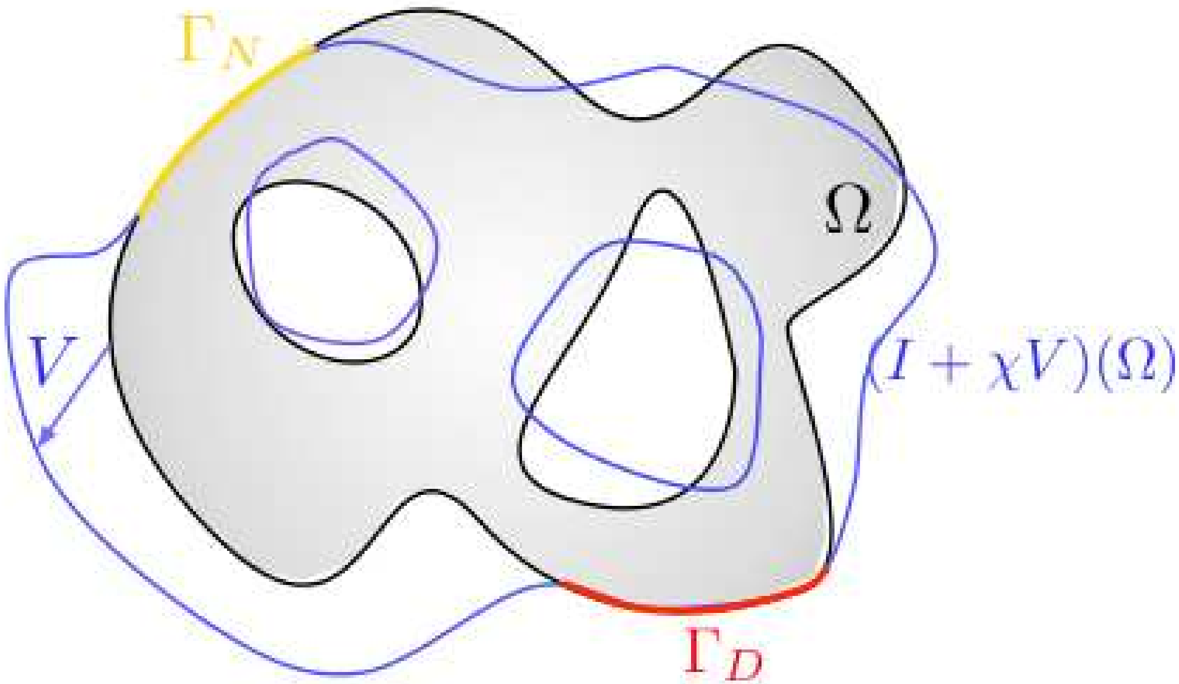
where p_Ω is the first adjoint state, defined by

$$\begin{cases} -\mathrm{div}(Ae(p_\Omega)) = -\nabla_u j(0, u_\Omega) & \text{in } \Omega, \\ p_\Omega = 0 & \text{on } \Gamma_D, \\ Ae(p_\Omega)n = 0 & \text{on } \Gamma \cup \Gamma_N. \end{cases}$$

If $\nabla_f j(0, u_\Omega) - p_\Omega \neq 0$ in $L^2(\Omega)^d$, then $\tilde{\mathcal{J}}$ is shape differentiable (with two additional adjoint states).

Second case: geometric uncertainties.

Perturbed shapes $(I+\chi V)(\Omega)$, $\quad V \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, $\;||V||_{L^\infty(\mathbb{R}^d)^d} \leq m$.



χ is a smooth localizing function such that $\chi \equiv 0$ on $\Gamma_D \cup \Gamma_N$.

Theorem.

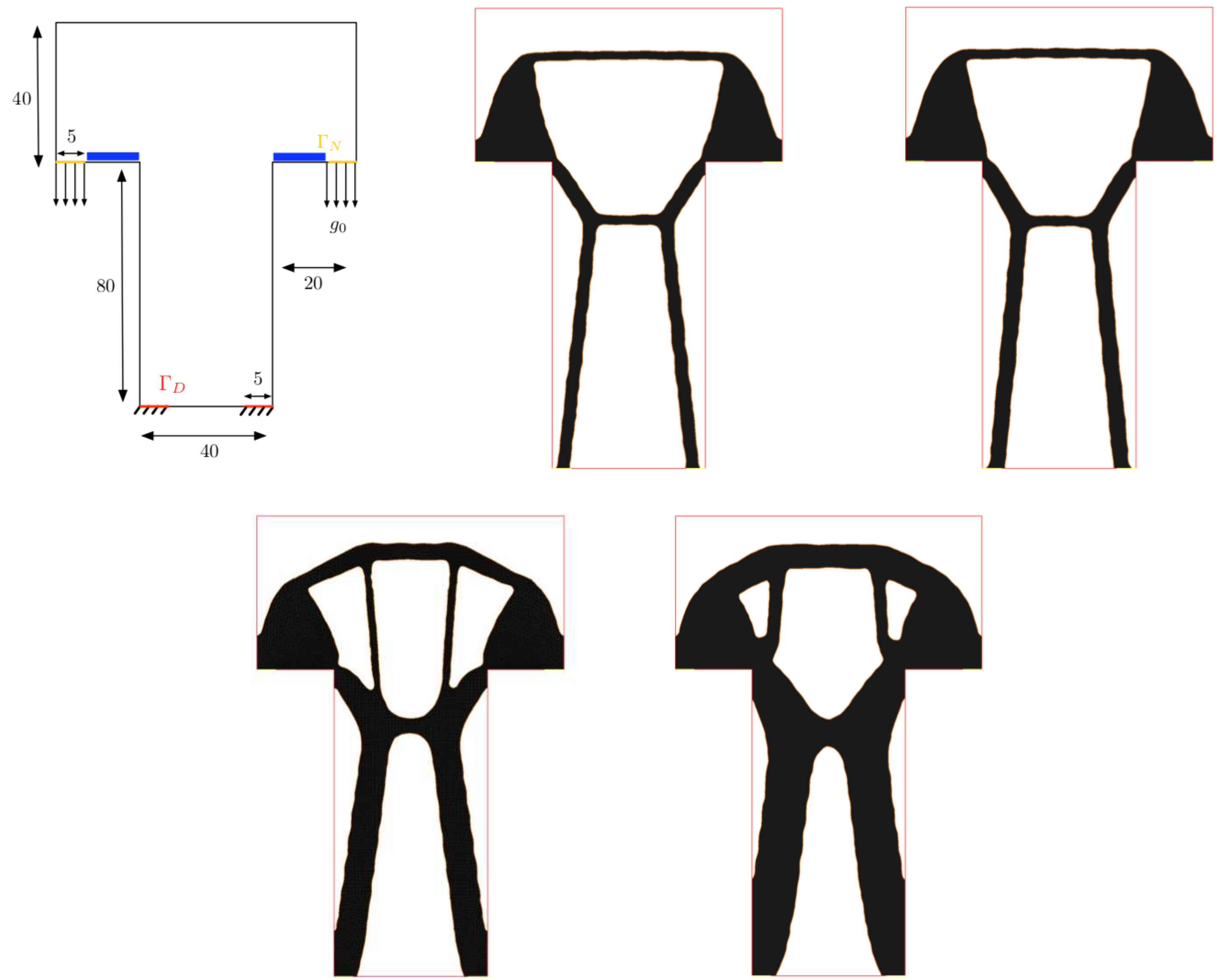
The linearized worst-case design objective function is

$$\widetilde{\mathcal{J}}(\Omega) = \int_\Omega j(u_\Omega)\,dx + m\int_\Gamma \chi \Big| j(u_\Omega) + Ae(u_\Omega) : e(p_\Omega) - f \cdot p_\Omega \Big|\, ds,$$

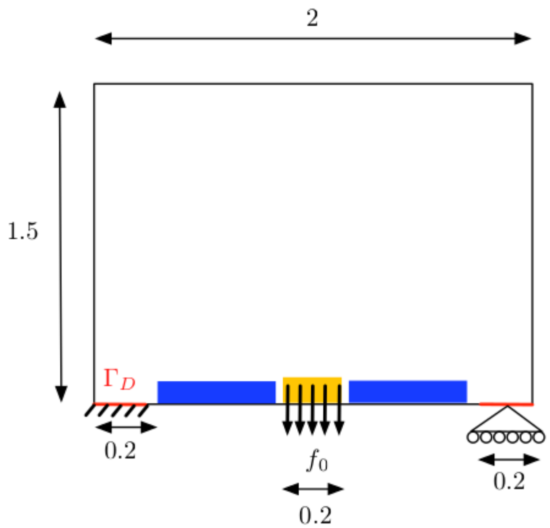
where p_Ω is the (previous) adjoint state.

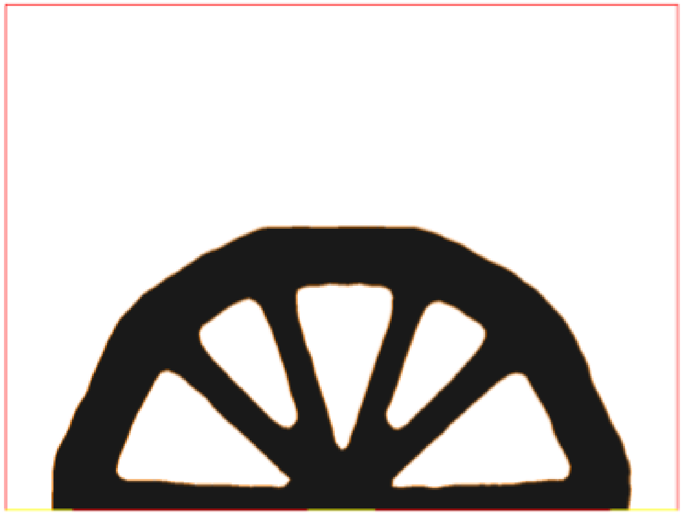
If $E_\Gamma := \{x \in \Gamma,\ \left(j(u_\Omega) + Ae(u_\Omega) : e(p_\Omega) - f \cdot p_\Omega\right)(x) = 0\}$ has zero Lebesgue measure, then it admits a (hugly) shape derivative $\widetilde{\mathcal{J}}'(\Omega)(\theta)$ involving two (new) additional adjoints q_Ω, z_Ω.

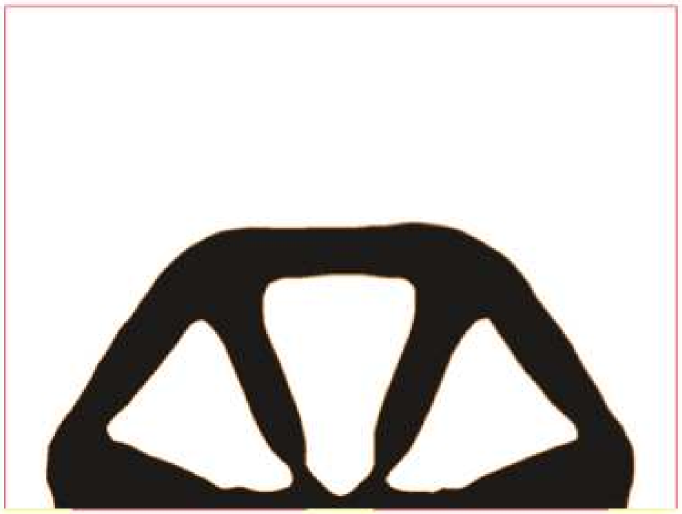
Load uncertainties in geometric optimization (compliance)



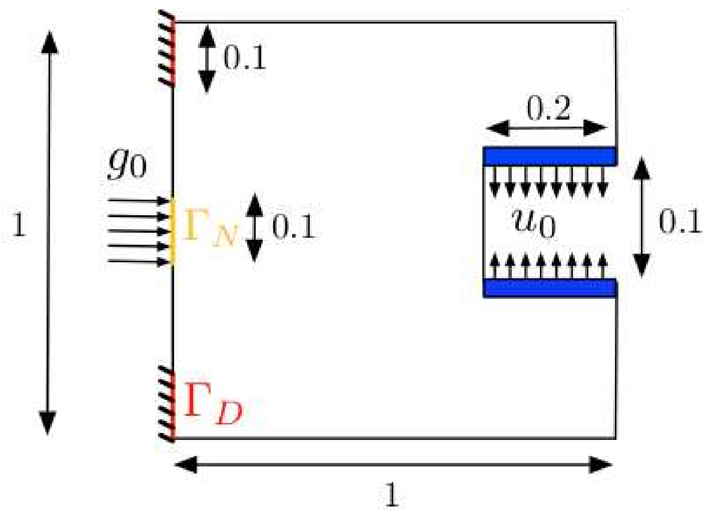
Load uncertainties in geometric optimization (compliance)

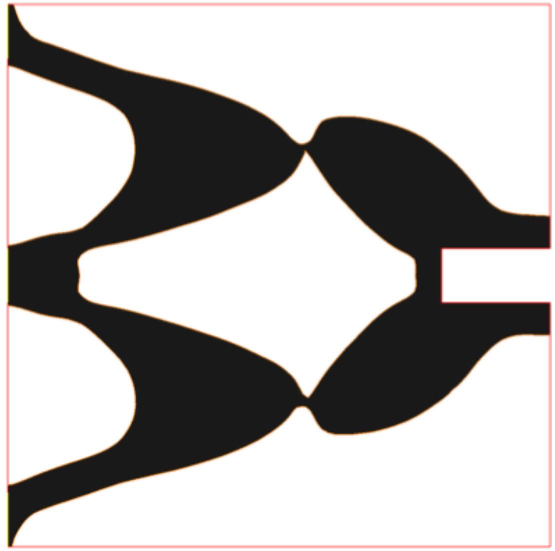


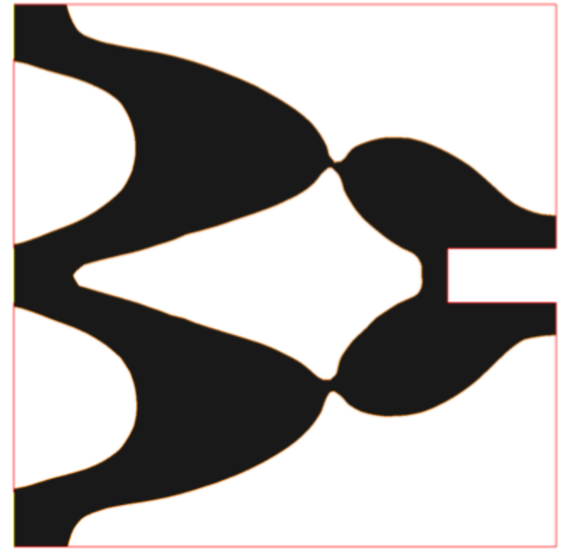


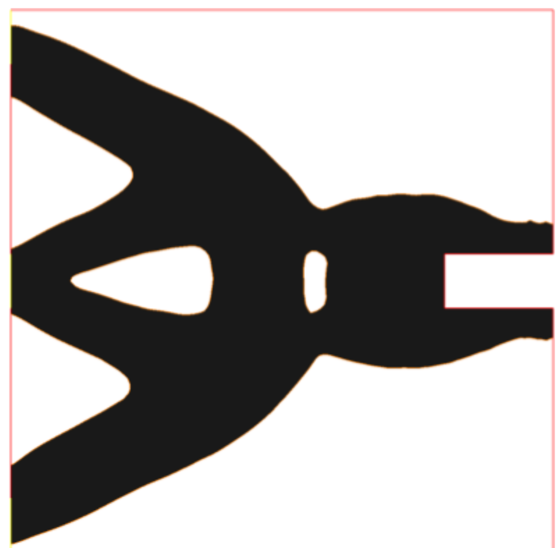


Geometric uncertainties in geometric optimization









Geometric uncertainties (stress minimization)