Plan

Quelques résultats récents sur des problèmes inverses de sources : reconstruction et stabilité.

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4 novembre 2014

Plan

Outline

1 Some inverse source problems

2 Inverse EEG problem
- Identification : Iterative methods
- Identification : Semi-Iterative methods

3 Sources with small supports : Algebraic method

4 Identification algorithm

5 Stability : monopoles and dipoles

6 Numerical results : dipole sources

Outline

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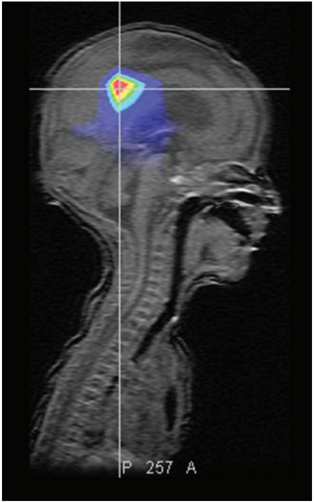
Motivation

One motivation consists in determining locations of epileptic foci.



Outline

The time-harmonic full Maxwell equations

Let Ω (the head) be a bounded domain in $\mathbb{R}^3$ with boundary Γ. In biomedical, the epilepsy is often caused by electrical discharges originate from a small volume and approximated by a current dipole $q.\delta_S$. Then the resulting electromagnetic field E, H is governed by

$$\begin{aligned} \nabla \times E &= -i\omega\mu H && \text{in} \quad \mathbb{R}^3 \\ \nabla \times H &= i\omega\epsilon E + \sigma E + J && \text{in} \quad \mathbb{R}^3 \end{aligned}$$

$$J = \sum_1^m q_j.\delta_{S_j}$$

σ, ϵ and μ are the electromagnetic parameters of Ω. The inverse problem consider consists in determining $m,\ S_j,\ q_j$ from the measurements

$$(f := E \times n_{|\Gamma}\ ,\ g := H \times n_{|\Gamma})$$

H. Ammari, 2002 SIAM ; S. He, V. Romanov 98 W. M. ; R. Albanese, P. Monk 2006 IP ; AEB ,T. Nara 2013 IP

EEG/MEG

- MEG : The static (or quasi-static) Maxwell's equations :

$$\begin{aligned} \nabla \times E &= 0 \ (\cancel{-i\omega\mu H}) \\ \nabla \times H &= \cancel{i\omega\epsilon E} + \sigma E + J \end{aligned}$$

where J denotes de primary cerebral current

Inverse MEG problem : Determine J from H

AEB T. Nara, 2010, Inverse Ill-Posed Probl.

- EEG : Maxwell's equations became :

$$\begin{aligned} -\nabla.(\sigma \nabla u) &= \nabla.J \quad \text{in} \quad \Omega \\ \sigma \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

Inverse EEG problem : The inverse EEG problem consists

in determining J from $u = f$ given on a part of the boundary $S \subset \Gamma$ (AEB T. HD, 2000 IP, thèse Maha Farah 2007,.....)

The eddy current equations

It is obtained by by neglecting the displacement current term :

$$\begin{aligned} \nabla \times E &= -i\omega\mu H && \text{in } \mathbb{R}^3 \\ \nabla \times H &= \cancel{i\omega\epsilon E} + \sigma E + J && \text{in } \mathbb{R}^3 \end{aligned}$$

$$J = \sum_{1}^{m} \vec{q_j}.\delta_{S_j}$$

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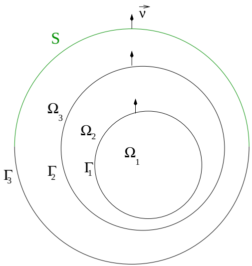
$$(f := E \times n_{|\Gamma}\ ,\ g := H \times n_{|\Gamma})$$

Ana Alonso Rodrguez, Jessika Camano and Alberto Valli, 2012 IP

Outline

The considered medium



- The domain Ω is assumed to be partitioned into non-overlapping sub-domains $\Omega_j,\ j = 1, ..., r.$
- In Ω_1 the conductivity σ is constant : $\sigma = \sigma_1$.
- $\partial\Omega_1 = \Gamma_1,\ \partial\Omega_j = \Gamma_j \cup \Gamma_{j-1}, j = 2, ..., r$ and $\Gamma_r = \Gamma$.
- $\Gamma_j \cap \Gamma_{j-1} = \emptyset$ where Γ_j is the interface between domains Ω_j and Ω_{j+1}.
- $S_j \in \Omega_1$.

Inverse EEG problem

Let $F = \sum_{k=1}^{m} \mathbf{q}_k . \nabla \delta_{S_k}$. If the number m, the locations S_j and the moments q_j are given, then the problem

$$\begin{aligned} -\nabla.(\sigma \nabla u) &= F \quad \text{in} \quad \Omega \\ \sigma \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

is well posed (up a constant) for which the trace $u_{|S}$ is well defined in $H^{\frac{1}{2}}(S)$ and then one can define the observation operator

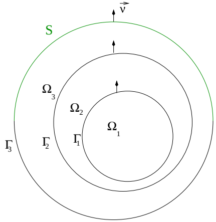
$$B[F] := u_{|S}$$

where $S \subset \Gamma$

Iterative method : Kohn-Vogelius method

- The idea of this method is :

  to minimize over Ω the energy difference between the solutions of two forward problems. One of them has a Neumann condition on Γ and the second one has a Dirichlet condition on S based on the measured data f.



- Bibliography :
  - *R.V. Kohn & M. Vogelius*, 1987.
  - *R.V. Kohn & A. Mckenney*, 1990.
  - ...

Kohn-Vogelius cost function

Forward Neumann problem :

$$\begin{aligned} -\nabla\cdot(\sigma\nabla u) &= F \quad \text{in} \quad \Omega \\ \sigma\frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma \end{aligned}$$

Forward mixed problem :

$$\begin{aligned} -\nabla\cdot(\sigma\nabla v) &= F \quad \text{in} \quad \Omega \\ v &= f \quad \text{on} \quad S \\ \sigma\frac{\partial v}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma\setminus S \end{aligned}$$

We suppose m known, denote $\varphi = (q_j, S_j)$ and set $\psi = u - v$.

- Definition :

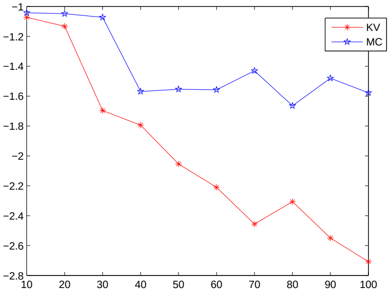
$$J_f(\varphi) = \frac{1}{2}\int_\Omega \sigma\,|\nabla\psi|^2 dx + \frac{1}{2}\int_S |\psi|^2 ds$$

- The Green formula gives :

$$J_f(\varphi) = \frac{1}{2}\int_S \psi.\sigma\frac{\partial\psi}{\partial\nu} ds + \frac{1}{2}\int_S |\psi|^2 ds$$

The data f is known partially on the scalp $S \subset \Gamma_3$

- Three dipoles



Euclidean relative error $\frac{|\varphi_{ex} - \varphi_c|}{|\varphi_{ex}|}$ in the case of the surface of data S covers respectively 10%, 20%, $\cdots$, 100% of the surface of scalp Γ_3.

Semi-Iterative methods

The identification process is decomposed into two parts :

First step : Consists in solving Cauchy problems in $\Omega \setminus \bar{\Omega}_1$

$$f_{|S} \longrightarrow (\rho, \theta) = (u, \sigma \frac{\partial u}{\partial \nu}) \text{ on } \Gamma_1$$

Second step :

$$\begin{aligned} -\sigma \triangle u &= F && \text{in } \Omega_1 \\ \sigma \frac{\partial u}{\partial \nu} &= \theta && \text{on } \Gamma_1 \\ u &= \rho && \text{on } \Gamma_1 \end{aligned}$$

with

$$F = \sum_{k=1}^{m} q_k . \nabla \delta_{S_k} \text{ or } F = \sum_{k=1}^{m} \lambda_k \delta_{S_k}$$

or F is a finite linear combination of multipolar sources.

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Outline

Problem statement

Our objective consists in determining the source

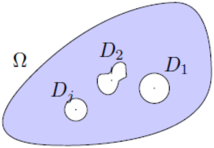
$$F = \sum_{j=1}^{N} h_j \chi_{D_j} \quad \text{with} \quad D_j = S_j + \varepsilon B_j$$

considering Helmholtz's equation

$$\Delta u + k^2 u \;=\; F \quad \text{in} \quad \Omega,$$

and the boundary measurements

$$(f, g) = (u_{|\Gamma}, \frac{\partial u}{\partial n}_{|\Gamma})$$



Problem statement

where

1. $B_j \subset \mathbb{R}^3$ are bounded domains containing the origin.
2. The points $S_j = (x_j, y_j, z_j)$ are assumed to be mutually distinct.
3. ϵ is a positive real number less than 1.
4. Their densities h_j are functions belonging to the space $L^2(\Omega)$.

Precisely, we are interested in determining the number N, the locations S_j and some characteristics of the domains D_j, for example, their masses and their moments.

One will see that solving this problem is equivalent, modulo ϵ, to solve an inverse multipolar source problem .

Inverse Problem

- $F = \sum_{j=1}^{N} h_j \chi_{D_j}$ with $D_j = S_j + \varepsilon B_j$
- ISP consists in determining F from the Cauchy data

$$(f, g) = (u_{|\Gamma}, \frac{\partial u}{\partial n}_{|\Gamma})$$

where three questions are considered :

1. *Uniqueness* : Does (f, g) uniquely determine F : $N,\ h_j,\ S_j,\ B_j$?
2. *Stability* : How does ~~N~~, $S_j,\ h_j,\ B_j$ depend on (f, g) ?
3. *Reconstruction method* : Are there constructive methods for determining $N,\ h_j,\ S_j,\ B_j$?

Outline

Principle of the method

Let u be the solution to Helmholtz's equation

$$\Delta u + k^2 u \quad = \quad F \quad \text{in} \quad \Omega.$$

First we introduce the test functions v

$$H_k = \{v \in H^1(\Omega) : \triangle v + k^2 v = 0\}$$

then we define the operator

$$\mathcal{R}(g, f, v) = -\int_\Gamma (f \frac{\partial v}{\partial \nu} - g v) ds$$

and then, one has

$$\mathcal{R}(g, f, v) = \sum_{j=1}^{N} \int_{D_j} h_j(X) v(X) dX$$

Principle of the method

How to choose v that would allow us to determine

$$N,\ h_j,\ S_j,\ B_j$$

Using, for all $n \in \mathbb{N}$, the functions

$$v_n^a(x,y,z) = (x+iy)^n e^{\pm ikz}$$

lead to the relationships between the source F and (f,g)

$$\mathcal{R}(v_n^a, f, g) = \sum_{j=1}^{N} \sum_{\beta=0}^{n} \nu_j^{\beta,a} \binom{n}{\beta} (P_j^a)^{n-\beta}, \quad \text{for all} \quad n \in \mathbb{N}$$

$$\nu_j^{\beta,a} = \varepsilon^{3+\beta} e^{\pm ikz_j} \int_{B_j} \tilde{h}_j(t)[t_1 + it_2]^\beta e^{\pm ik\varepsilon t_3} dt$$

$$\tilde{h}_j(t) = h_j(S_j + \varepsilon t)$$

Sources of small supports

For a given positive $\varepsilon < 1$, we choose a fixed integer K such that ε^{K+4} is small enough and we set

$$\alpha_n^a := \sum_{j=1}^{N} \sum_{\beta=0}^{K} \nu_j^{\beta,a} \binom{n}{\beta} (P_j^a)^{n-\beta}, \quad \text{for all} \quad n \in \mathbb{N}.$$

- For $n \leq K$, $\mathcal{R}(v_n^a, f, g) = \alpha_n^a$
- For $n > K$, $\mathcal{R}(v_n^a, f, g) = \alpha_n^a + O(\varepsilon^{K+4})$

Remark

When $k = 0$, $\nu_j^{0,a}$ correspond to the mass of the domain D_j and $\nu_j^{1,a}$ correspond to the projection of its moments onto the xy-plane.

Principle of the method

Moreover, using

$$v_n^b = (y+iz)^n e^{\pm ikx} \quad \text{and} \quad v_n^c = (x+iz)^n e^{\pm iky}$$

we get, the following relationships

$$\alpha_n^r = \sum_{j=1}^{N} \sum_{\beta=0}^{K} \nu_j^{\beta,r} \binom{n}{\beta} (P_j^r)^{n-\beta} \quad \text{for} \quad r = a, b, c$$

where P_j^a, P_j^b and P_j^c are the projections of points S_j onto the xy, yz- and xz-complex planes respectively.

Remark

The same relationships can be obtained for multipolar sources

$$J = \sum_{j=1}^{N} \sum_{\alpha=0}^{K} \lambda_j^{\{\alpha_1,\alpha_2,\alpha_3\}} \frac{\partial^\alpha}{\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3}} \delta_{\mathbf{S_j}}, \quad \alpha = \alpha_1 + \alpha_2 + \alpha_3$$

Hankel matrix

Let $\bar{J}$ be an upper bound of $(K+1)N$, consider the matrix

$$H^r_{\bar{J}} = \begin{pmatrix} \alpha^r_0 & \alpha^r_1 & \cdots & \alpha^r_{\bar{J}-1} \\ \alpha^r_1 & \alpha^r_2 & \cdots & \alpha^r_{\bar{J}} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^r_{\bar{J}-1} & \alpha^r_{\bar{J}} & \cdots & \alpha^r_{2\bar{J}-2} \end{pmatrix}$$

Lemma (B. Abdelaziz, AEB, A. El Hajj, 2014)

$\exists$ A^r_0 *of size* $\bar{J} \times (K+1)N$, $\exists$ I^r *of size* $(K+1)N \times (K+1)N$ *such that*

$$H^r_{\bar{J}} = A^r_0 I^r (A^r_0)^t$$

- I^r is a multi-diagonal matrix.
- I^r is invertible if and only if $\nu^{K,r}_j \neq 0$ for $j = 1, ..., N$.
- A^r_0 is a Vandermonde matrix type.

Matrix I^r

$$\bar{I}^r = \begin{pmatrix} \nu^{0,r} & \nu^{1,r} & \cdots & \nu^{K,r} \\ \vdots & \vdots & \iddots & \vdots \\ \nu^{K-1,r} & \nu^{K,r} & \cdots & 0 \\ \nu^{K,r} & 0 & \cdots & 0 \end{pmatrix}$$

where

$$\nu^{\beta,r} = \begin{pmatrix} \nu_1^{\beta,r} & 0 & \cdots & 0 \\ 0 & \nu_2^{\beta,r} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu_N^{\beta,r} \end{pmatrix}$$

Number of sources

Theorem (B. Abdelaziz, AEB, A. El Hajj, 2014)

Let $\bar{J}$ *is a known upper bound of* $(K+1)N$. *Let* $H^r_{\bar{J}}$ *be the Hankel matrix*

$$H^r_{\bar{J}} = \begin{pmatrix} \alpha^r_0 & \alpha^r_1 & \cdots & \alpha^r_{\bar{J}-1} \\ \alpha^r_1 & \alpha^r_2 & \cdots & \alpha^r_{\bar{J}} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha^r_{\bar{J}-1} & \alpha^r_{\bar{J}} & \cdots & \alpha^r_{2\bar{J}-2} \end{pmatrix} \quad r = a, b, c$$

If the projected points P^r_j *are distinct mutually , then*

$$rank\left(H^r_{\bar{J}}\right) = (K+1)N \quad \text{if and only if} \quad \nu^{K,r}_j \neq 0 \; j = 1, ..., N.$$

Identification of S_j

Theorem (B. Abdelaziz, AEB, A. El Hajj, 2014)

Let $p = (K+1)N$ *and* $\xi_p^r = (\alpha_p^r, ..., \alpha_{2p-1}^r)^t$.
Let $C^r = (c_0^r, ..., c_{p-1}^r)^t$ *be the solution of* $H_p^r C^r = \xi_p^r$ *and* B^r *be the companion matrix*

$$B^r = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ c_0^r & c_1^r & \cdots & \cdots & c_{p-1}^r \end{pmatrix}$$

If P_j^r *are distinct and* $\nu_j^{K,r} \neq 0$, $j = 1, ..., N$, *then*

1. B^r *admits* N *eigenvalues of multiplicity* $(K+1)$.
2. *The* N *eigenvalues are the projections* P_j^r.

Outline

Stability of monopolar and dipolar sources

We need to introduce some notations and precise statement of additional information.

$$\alpha = \min_{1 \leq j \leq N} \{d(\Gamma, S_j)\},$$

consider

$$\Omega_\alpha = \{x \in \Omega \; / \; d(\Gamma, x) > \alpha\},$$

and set

$$\beta = diam(\Omega) - \alpha.$$

Let $S_j = (x_j, y_j, z_j)$, $P_j = x_j + iy_j$ and $Q_j = y_j + iz_j$

$$\varrho_1 = \min_{1 \leq j,n \leq m, j \neq n} \|P_j - P_n\|, \; \varrho_2 = \min_{1 \leq j,n \leq m, j \neq n} \|Q_j - Q_n\|$$

and set

$$\varrho = \min(\varrho_1, \varrho_2).$$

Stability

Let now

$$P^\ell = (P_j^\ell)_{1\leq j\leq m^\ell}, \ell = 1, 2$$

be two configurations such that

$$\mathbf{P}^\ell \subset \Omega_\alpha, \ \ell = 1, 2$$

Consider the Hausdorff distance between the configurations P^1 and P^2 defined as follows

$$d_H(P^1, P^2)$$

$$=$$

$$\max\left[\max_{1\leq\ell\leq m^2}\ \min_{1\leq j\leq m^1}\|P_\ell^2 - P_j^1\|,\ \max_{1\leq\ell\leq m^1}\ \min_{1\leq j\leq m^2}\|P_\ell^1 - P_j^2\|\right].$$

Stability : case $J = \sum_{j=1}^{m} \lambda_j \delta_{S_j}$

Theorem (AEB, A. El Hajj IP 2013)

Let u^ℓ *for* $\ell = 1, 2$ *be the solutions of Helmholtz's equation corresponding to the sources* $J^\ell := \sum_{j=1}^{m^\ell} \lambda_j^\ell S_j^\ell$

Let $(f^\ell, g^\ell) := (u^\ell_{|\Gamma}, \frac{\partial u^\ell}{\partial \nu}_{|\Gamma})$, $\ell = 1, 2$. *Then,*

$$\boxed{d_H(S^1, S^2) \leq 2 \max_{\ell=1,2} \left[\frac{\beta^{m^1+m^2-1}}{\varrho^{m^{(3-\ell)}-1}} \frac{\sqrt{|\Gamma|}}{c_1} [\varepsilon_1 + c_2 \varepsilon_2] \right]^{\frac{1}{m^\ell}}}$$

$$\varepsilon_1 = \|g^2 - g^1\|_{L^2(\Gamma)}, \quad \varepsilon_2 = \|f^2 - f^1\|_{L^2(\Gamma)},$$

$$0 < cte < c_1 = \min_{\substack{1 \leq j \leq m^1 \\ 1 \leq n \leq m^2}} (|\lambda_j^1|, |\lambda_n^2|) \quad c_2 = \sqrt{2\frac{(m^1 + m^2 - 1)^2}{\beta^2} + k^2}.$$

Stability

Theorem (AEB, A. El Hajj IP 2013)

Let, u^ℓ for $\ell = 1,2$ be the solutions of Helmholtz's equation corresponding to the sources $J^\ell := \sum_{j=1}^m \lambda_j^\ell S_j^\ell$. Let $(f^\ell, g^\ell) := (u^\ell_{|\Gamma}, \frac{\partial u^\ell}{\partial \nu}_{|\Gamma})$ for $\ell = 1,2$. Then, there exists a permutation π of the integer $1, \cdots, m$, such that

$$\max_{1\le j\le m} \|S_j^2 - S^1_{\pi(j)}\| \le 2\frac{\beta^2}{\varrho}\left[\frac{\sqrt{|\Gamma|}}{c_1}\frac{\varrho}{\beta}\left[\varepsilon_1 + c_2\varepsilon_2\right]\right]^{\frac{1}{m}}$$

$$\varepsilon_1 = \|g^2 - g^1\|_{L^2(\Gamma)}, \quad \varepsilon_2 = \|f^2 - f^1\|_{L^2(\Gamma)}$$

$$0 < cte < c_1 = \min_{\substack{1\le j\le m\\ \ell=1,2}} |\lambda_j^\ell|, \quad c_2 = \sqrt{2\frac{(2m-1)^2}{\beta^2} + k^2}.$$

Stability

In order to establish our stability estimate on point sources, we need to recall the following theorem, borrowed from graph theory and called Hall-Rado Theorem.

Theorem (Hall-Rado)

Consider an even graph having $2m$ points

$$A = (a_1, \cdots, a_m), \quad B = (b_1, \cdots, b_m)$$

We connect some pairs (a_i, b_j) such that : For every $k \in \{1, \cdots, m\}$ and every $(a_{j_1}, \cdots, a_{j_k})$ of A, at least k elements b_j of B are connected to one of them.

Then, there exists a permutation π of the integer $1, \cdots, m$ such that :

$$a_j \text{ is connected to } b_{\pi(j)} \text{ for every } j.$$

Stability

This theorem has an interpretation as a solution of the problem of marriage posed in the following terms that we borrowed from

> P. R. Halmos and H. E. Vaughan, The marriage problem, Amer. J. Math., 72 (1950), pp. 214-215.

Suppose that each of a (possibly infinite) set of boys is acquainted with a finite set of girls. Under what conditions is it possible for each boy to marry one of his acquaintances ? It is clearly necessary that every finite set of k boys be, collectively, acquainted with at least k girls ; this condition is also sufficient.

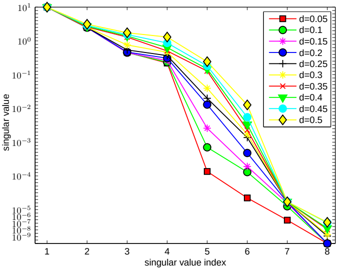
Outline

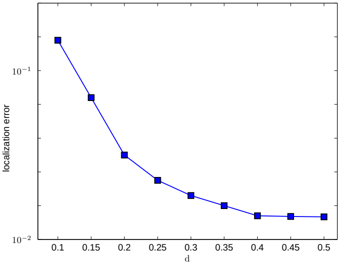
What happens when the number is wrongly-estimated ?

truncation level	Estimated 2D Positions
8	0.808 + 1.082i
	-1.161 - 0.783i
	-0.5995 - 0.400i
	0.500 + 0.500i
	0.6000 - 0.3000i
5	-0.689 - 0.418i
	-0.461 - 0.409i
	0.685 - 0.410i
	0.499 + 0.652i
	0.727 + 0.312i

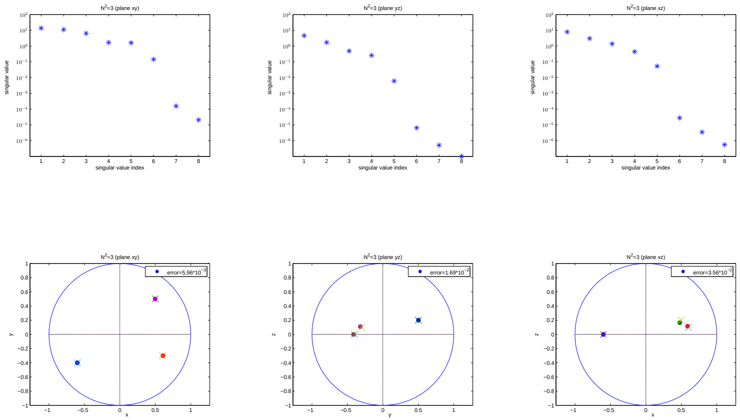
j (location ♯)	1	2	3
S_j	(0.6,-0.3,0.1)	(-0.6,-0.4,0.0)	(0.5,0.5,0.2)

Effect of separability



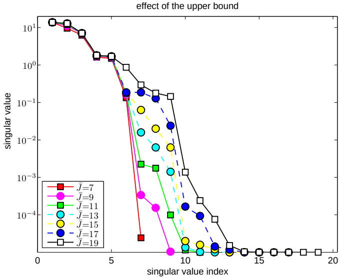


Obtaining the 3D coordinates



The impact of the upper bound $\overline{J}$

$\overline{J}$	7	9	11	13	15	17	19
$\Vert\delta H^a_{\overline{J}}\Vert_F \simeq$	0.62	0.8	0.97	1.15	1.33	1.50	1.68



Thank You For Your Attention